Pergamon

Int. J. Heat Mass Transfer. Vol. 38, No. 3, pp. 403–408, 1995

0017-9310/95 \$9.50+0.00

0017-9310(94)00209-6

Extended Leveque solution for laminar heat transfer to power-law fluids in pipes with wall slip

YEN-PING SHIH, CHUNG-CHING HUANG and SUN-YUAN TSAY

Department of Chemical Engineering, National Cheng Kung University, Tainan, Taiwan, 700, Republic of China

(*Received for publication in final form* 21 *July* 1994)

Abstract—The effect of wall slip on the laminar heat transfer to power-law fluids in pipes with constant wall temperature and constant wall heat flux is investigated for the case of a short axial distance. The transformed temperature is expanded in a Leveque series in terms of the powers of $s^{-1/2}$, where s is the Laplace transformation variable with respect to the axial distance. For a constant wall temperature, the zero-order solution is the Leveque solution, whereas, for a constant wall heat flux, the first-order solution is the Leveque solution. Numerical solutions are obtained to show the convergence as well as the effect of wall slip on the Nusselt number vs the Graetz number. A significant increase in the Nusselt number is shown with an increase in the wall slip velocity. The effect of the Brinkman number is less important for high Graetz numbers.

INTRODUCTION

The transport processes of polymer solution and melts with wall slip phenomena were summarized by Mashelkar and Dutta [1]. Ju and Chou [2] analyzed the effect of wall slip on the laminar heat transfer of power-law fluids in pipes with a constant wall temperature by the method of eigenfunction expansion. Due to the nature of the method, the solutions are rapidly convergent for small Graetz numbers (i.e. for a distance far from the thermal entrance of the pipes). The wall slip was expressed as a ratio of the slip velocity to the maximum velocity at the pipe center.

Sparrow and Lin [3] analyzed the laminar heat transfer of rarefied gases in pipes under the wall slip condition by assuming that the slip velocity was proportional to the shear stress at the wall. This expression of the wall slip condition was also adapted by Sparrow *et al.* [4] and Singh and Laurence [5]. Since, in pipe flow, shear stress is uniform, a constant slip velocity is obtained along the pipe.

Another approach was used by Michaeli [6] in a study of the wall slip condition in polymer processing. The wall slip condition is based on a force balance at the wall between the pressure force and the friction force of a fluid element. The velocity distribution varies from the pipe entrance to the exit, where the velocity distribution becomes plug flow. At the pipe entrance, there is no wall slip.

The entrance heat transfer problem using a linear velocity profile is known as the Leveque solution [7], which is an asymptotic solution for short axial distances. An extension of the Leveque solution was made by Newman [8] with a transformed coordinate perturbation method for mass transfer of Newtonian fluids. The concentration distribution of the entrance laminar mass transfer with constant wall concentration is expanded into a power series of a transformed axial distance. The zero-order solution is the Leveque solution. The method was extended further by Shih and Tsou [9] to power-law fluids with viscous dissipation for constant wall temperature as well as constant heat flux. The same problem was also studied by Richardson [10]. Another analytical technique for the same problem was investigated by Gottifredi *et al.* [11], Gottifredi and Flores [12] and Chen and Ju [13]. Using a Laplace transformation with respect to the axial coordinate of the normalized temperature Θ, the variable $sL(\Theta)$ (where s is the Laplace transformation variable) is expanded into a power series of $s^{-1/3}$ after a suitable coordinate transformation.

In this paper, the effect of wall slip, which is expressed as a slip parameter, on the entrance laminar heat transfer of power-law fluids in circular pipes is investigated. This study is restricted to short axial distances with constant wall temperature as well as constant wall heat flux.

GOVERNING EQUATIONS

The power-law fluid model is described by [7]

$$\tau_{RZ} = -m\left|\frac{\mathrm{d}V}{\mathrm{d}R}\right|^{(1/N)-1}\frac{\mathrm{d}V}{\mathrm{d}R} \quad N = \frac{1}{n} \tag{1}$$

where τ_{RZ} is the Z-momentum transferred in the R-direction, $V(R)$ is the velocity direction, R is the radial coordinate from the center of the pipe, m is a parameter and n the flow index. The fully developed lami-

NOMENCLATURE

A	parameter defined by equation (9d)
B	parameter defined by equation (9e)
Br	Brinkman number defined by equations (12) and (14)
Gz	Graetz number defined by equations (36)
h	heat transfer coefficient
k	thermal conductivity of fluid
$L[.]$	Laplace transformation
m	parameter of power-law fluids defined by equation (1)
n	flow index of power-law fluids
N	$1/n$
Nu	local Nusselt number defined by equations (34) and (37)
Q	volumetric flow rate in the pipe
q	wall heat flux, a constant
R	radial distance from the center of the pipe
R_o	radius of the pipe
r	R/R_o
s	Laplace transformation variable
$T(R,Z)$	temperature of the fluid
T_o	inlet fluid temperature, a constant
T_w	wall temperature, a constant
$V(R)$	velocity in the axial direction
V_m	maximum velocity in the pipe
V_s	slip velocity at the wall
$\langle V\rangle$	average velocity in the pipe
x	transformed radial coordinate defined by equation (24)
Z	axial distance of the thermal entrance section
z	normalized axial distance defined by equation (9b).

Greek symbols

β	$V_s/\langle V\rangle$, slip parameter
$\Theta(r,z)$	dimensionless temperature defined by equations (11) and (13)
$\Phi(r,s)$	$sL[\Theta]$
$\Phi(x,s)$	transformed temperature
$\Phi_i(x)$	coefficients of the power series of $\Phi(x,s)$ defined by equation (28).

nar velocity distribution of a power-law fluid flowing in a circular pipe of radius R_o with slip velocity V_s is expressed as

$$V = V_s + \frac{N+3}{N+1}(\langle V\rangle - V_s)\left[1-\left(\frac{R}{R_o}\right)^{N+1}\right] \quad (2)$$

where $\langle V\rangle$ is the average velocity:

$$\langle V\rangle = \frac{1}{\pi R_o^2}\int_0^{R_o} 2\pi RV\,\mathrm{d}R. \quad (3)$$

Neglecting the axial conduction and the variations of the physical properties with temperature, the energy equation of the thermal entrance expressed in the temperature distribution $T(R,Z)$ for laminar flow becomes

$$\rho C_p V\frac{\partial T}{\partial Z} = \frac{k}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) + m\left|\frac{\mathrm{d}V}{\mathrm{d}R}\right|^{(1/N)-1}\left(\frac{\partial V}{\partial R}\right)^2 \quad (4)$$

where ρ, C_p and k are, respectively, the density, heat capacity and thermal conductivity of the fluid. The second term on the right-hand side of equation (4) represents the viscous dissipation.

For a constant inlet fluid temperature:

$$T(R,0) = T_o. \quad (5)$$

Symmetry at the center of the pipe gives

$$\left.\frac{\partial T}{\partial R}\right|_{R=0} = 0 \quad Z \geqslant 0. \quad (6)$$

For a constant wall temperature of T_w,

$$T(R_o, Z) = T_w \quad (7)$$

whereas, for a constant wall heat flux of q,

$$k\left.\frac{\partial T}{\partial R}\right|_{R=R_o} = q \quad Z \geqslant 0. \quad (8)$$

Equations (4)–(8) are normalized by defining dimensionless coordinates and parameters:

$$r = \frac{R}{R_o} \quad (9a)$$

$$z = \frac{kZ}{\rho C_p\langle V\rangle R_o^2} \quad (9b)$$

$$\beta = \frac{V_s}{\langle V\rangle} \quad (9c)$$

$$A = \frac{N+3-2\beta}{N+1} \quad (9d)$$

$$B = \frac{(N+3)(1-\beta)}{N+1}. \quad (9e)$$

The slip parameter β is defined as the ratio of the wall slip velocity to the mean velocity. It is consistent with that given by Sparrow and Lin [4]. The advantage of using β is that the expression of the Graetz number is explicitly free of the flow index and slip parameter, and is inversely proportional to the dimensionless axial distance. The relation between the slip parameter and the maximum velocity V_m is

$$\frac{1}{\beta} = \frac{\langle V\rangle}{V_s} = \frac{2}{N+3} + \frac{N+1}{N+3}\frac{V_m}{V_s} \quad (10)$$

where V_s/V_m is used by Ju and Chou [2]. For a constant wall temperature the dimensionless temperature $\Theta(r,z)$ and Brinkman number Br are defined as

$$\Theta = \frac{T-T_o}{T_w-T_o} \quad \text{CWT} \tag{11}$$

$$Br = \frac{m[(N+3)(\langle V\rangle - V_s)]^{(N+1)/N}}{k(T_w-T_o)R_o^{1/N-1}} \quad \text{CWT.} \tag{12}$$

For a constant wall heat flux, $\Theta(r,z)$ and Br are defined as

$$\Theta = \frac{k(T-T_o)}{qR_o} \quad \text{WHF} \tag{13}$$

$$Br = \frac{m[(N+3)(\langle V\rangle - V_s)]^{(N+1)/N}}{qR_o^{1/N}} \quad \text{WHF.} \tag{14}$$

Here CWT and WHF denote, respectively, constant wall temperature and constant wall heat flux. Therefore, equations (4)–(8) become

$$(A-Br^{N+1})\frac{\partial\Theta}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right)+Br\,r^{N+1} \tag{15}$$

with

$$\Theta(r,0) = 0 \tag{16}$$

$$\left.\frac{\partial\Theta}{\partial r}\right|_{r=0} = 0. \tag{17}$$

For a constant wall temperature

$$\Theta(1,z) = 1 \quad \text{CWT} \tag{18}$$

whereas, for a constant wall heat flux,

$$\left.\frac{\partial\Theta}{\partial r}\right|_{r=1} = 1 \quad \text{WHF.} \tag{19}$$

ANALYSIS

For a short axial distance, Laplace transformation of $\Theta(r,z)$ with respect to z and multiplication by the Laplace transformation variable s [11, 12] give

$$\Phi(r,s) = sL[\Theta] = s\int_0^\infty \exp(-sz)\Theta(r,z)\,dz. \tag{20}$$

Note that the Laplace transformation variable s has a dimension of the inverse of the normalized axial distance z. Using the entrance condition of equation (16), taking the Laplace transformation of $\Theta(r,z)$ in equation (15) and multiplying by s give an ordinary differential equation for $\Phi(r,s)$:

$$(A-Br^{N+1})s\Phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right)+Br\,r^{N+1}. \tag{21}$$

For a constant wall temperature and constant wall heat flux, equations (18) and (19) give, respectively,

$$\Phi(1,s) = 1 \quad \text{CWT} \tag{22}$$

$$\left.\frac{d\Phi}{dr}\right|_{r=1} = 1 \quad \text{WHF.} \tag{23}$$

Since the solutions are valid only at a short axial distance, the boundary condition at the pipe center [equation (17)] will not be used. Additional boundary conditions for the solution of the second-order differential equation of equation (21) will be discussed later.

Define a new coordinate variable x as

$$x = (1-r)s^{1/2}. \tag{24}$$

In terms of the transformed temperature distribution $\Phi(x,s)$ equations (21)–(23) become

$$\frac{d^2\Phi}{dx^2} = \frac{1}{s^{-1/2}-x}\frac{d\Phi}{dx}+(A-Br^{N+1})\Phi - Br(s^{-1}-xs^{1/2}) \tag{25}$$

and

$$\Phi(x,s) = 1, x = 0, \quad \text{CWT} \tag{26}$$

$$\left.\frac{d\Phi}{dx}\right|_{x=0} = -s^{1/2} \quad \text{WHF.} \tag{27}$$

A coordinate perturbation solution in the power of $s^{-1/2}$ is now assumed by letting

$$\Phi(x,s) = \Phi_0(x)+s^{-1/2}\Phi_1(x)+s^{-1}\Phi_2(x) + s^{-3/2}\Phi_3(x)+\cdots \tag{28}$$

This perturbation method is applicable for small $s^{-1/2}$, and higher-order terms can be neglected for small z. Substitution of equation (28) into equations (25)–(27) and comparison of the coefficients of the same power of $s^{-1/2}$ yield

$$\Phi_0'' = (A-B)\Phi_0$$

$$\Phi_1'' = (A-B)\Phi_1+B(N+1)x\Phi_0+\Phi_0'$$

$$\Phi_2'' = (A-B)\Phi_2-\frac{B(N+1)Nx^2}{2!}\Phi_0+x\Phi_0' + B(N+1)x\Phi_1+\Phi_1'-Br$$

$$\Phi_3'' = (A-B)\Phi_3+\frac{B(N+1)N(N-1)}{3!}x^3\Phi_0+x^2\Phi_0' - \frac{B(N+1)Nx^2}{2!}\Phi_1+x\Phi_1'+B(N+1)x\Phi_2 + \Phi_2'+Br(N+1)x$$

$$\Phi_4'' = (A-B)\Phi_4-\frac{B(N+1)N(N-1)(N-2)}{4!}x^4\Phi_0 + x^3\Phi_0'+\frac{B(N+1)N(N-1)}{3!}x^3\Phi_1 + x^2\Phi_1'-\frac{B(N+1)N}{2!}x^2\Phi_2+x\Phi_2' + B(N+1)x\Phi_3+\Phi_3'-\frac{Br(N+1)N}{2!}x^2 \tag{29}$$

where "′" means d/dx, for a constant wall temperature

$$\Phi_0(0)=1;\quad \Phi_i(0)=0,\quad i=1,2,3\ldots \quad \text{CWT} \tag{30}$$

and, for a constant wall heat flux,

$$\Phi_0'(0)=0;\quad \Phi_1'(0)=-1;$$

$$\Phi_i'(0)=0, i=2,3,\ldots \quad \text{WHF} \tag{31}$$

Another boundary condition for each equation of equations (29) is obtained by a boundary layer approximation. Let the temperature outside the thermal boundary layer equal the entrance temperature, i.e.

$$\Phi(\infty, s)=0. \tag{32}$$

Hence, for both cases

$$\Phi_i(\infty)=0,\quad i=0,1,2,\ldots \tag{33}$$

Analytical solutions can be obtained easily for $\Phi_0(x)$ and $\Phi_1(x)$, and may be obtained for $\Phi_2(x)$ with a certain effort. However, a numerical method is used to integrate equation (29) with appropriate boundary conditions.

Note that equation (28) is quite different from the case of no wall slip, in which expansion is to the power of $s^{-1/3}$. For the nonslip case, the variable x equals $(1-r)s^{1/3}$.

RESULTS OF CONSTANT WALL TEMPERATURE CASE

The local Nusselt number Nu is defined as

$$Nu=\frac{2R_o h}{k}=2\left.\frac{\partial\Theta}{\partial r}\right|_{r=1} \tag{34}$$

for a constant wall temperature, where h is the heat transfer coefficient. In terms of $\Phi_i(x)$

$$Nu=-\frac{2}{\sqrt{\pi}}[\Phi_0'(0)z^{-1/2}+\Phi_1'(0)\sqrt{\pi}+\Phi_2'(0)2z^{1/2}+\Phi_3'(0)\sqrt{\pi}z+\cdots]. \tag{35}$$

The Graetz number Gz is defined as

$$Gz=\frac{\rho C_p Q}{kZ}=\frac{\pi}{z} \tag{36}$$

where Q is the volumetric flow rate.

The zero-order solution $\Phi_0(x)$ which can be obtained analytically is known as the Leveque solution. The convergence of the extended Leveque solution is illustrated in Fig. 1 for the case of $n=0.25$, $Br=0$ and $\beta=0.1$. For a large Graetz number, i.e. for a short axial distance, the convergence is fast, as expected. For example, for a Graetz number of 1000, four or five terms in the series of equation (28) are enough. However, less terms are required for a large flow index n. Figures 2–5 show that the numerical results can be extended to lower Graetz numbers with an increase in the flow index.

Figures 2–5 show the effect of the slip parameter on

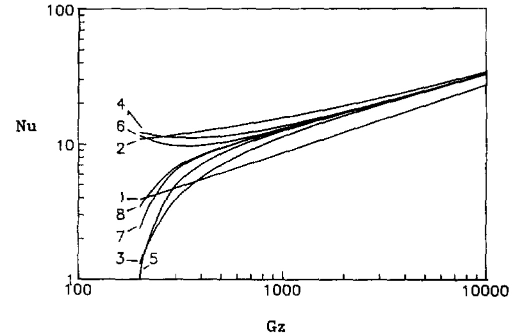

Fig. 1. Nusselt vs Graetz number for a constant wall temperature with $n=0.25$, $Br=0$ and $\beta=0.1$: 1, 2..., denote extended Leveque solutions with one, two..., terms, respectively.

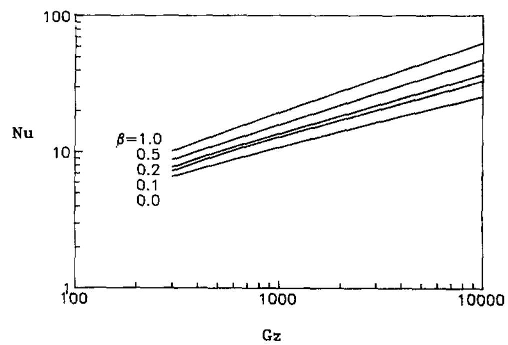

Fig. 2. Nusselt number vs Graetz number foa constant wall temperature with $n=0.25$, $Br=0$.

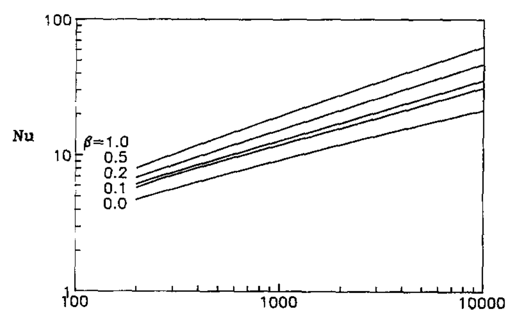

Fig. 3. Nusselt number vs Graetz number for a constant wall temperature with $n=0.5$, $Br=0$.

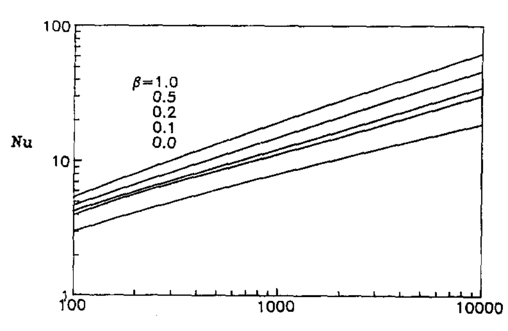

Fig. 4. Nusselt number vs Graetz number for a constant wall temperature with $n=1.0$, $Br=0$.

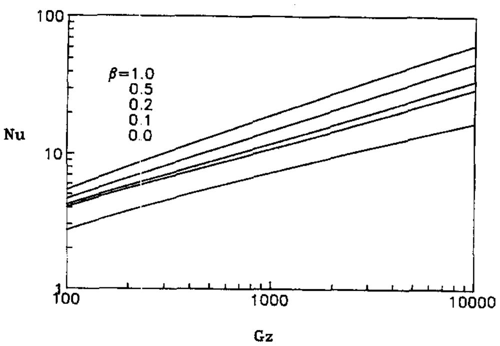

Fig. 5. Nusselt number vs Graetz number for a constant wall temperature with $n = 2.0$, $Br = 0$.

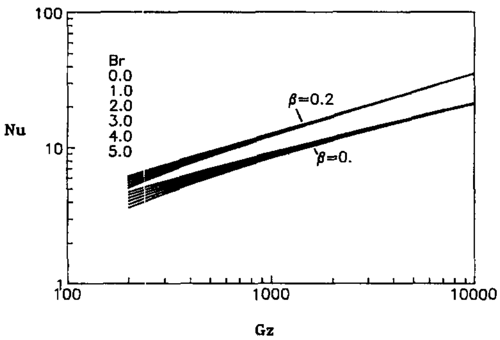

Fig. 6. Nusselt number vs Graetz number for a constant wall temperature with $n = 0.5$, $\beta = 0.2$ and $\beta = 0$.

the Nusselt number for the cases of $n = 0.25$, 0.5, 1 and 2, respectively. The Nusselt number increases with an increase in the slip parameter. This result is consistent with the eigenfunction expansion result of Ju and Chou [2]. The effect of the Brinkman number on the Nusselt number is minor, as shown in Fig. 6 for large Graetz numbers. Equations (29) show that $\Phi_0(x)$ and $\Phi_1(x)$ are independent of the Brinkman number.

RESULTS OF CONSTANT WALL HEAT FLUX CASE

For a constant wall heat flux, the local Nusselt number is expressed as

$$Nu = \frac{2R_o h}{k} = \frac{2}{\Theta(1,z)} = 2\left[\Phi_1(0)\frac{2}{\sqrt{\pi}}z^{-1/2} + \Phi_2(0)z + \Phi_3(0)\frac{1.333}{\sqrt{\pi}}z^{3/2} + \cdots\right]^{-1}. \quad (37)$$

Note that $\Phi_0(x) = 0$. The convergence of the local Nusselt number vs the Graetz number is shown in Fig. 7. About seven terms of the Leveque series are required for a Graetz number of about 600 and a flow index $n = 0.25$. However, less terms are required for large flow indices as well as for large Graetz numbers. In other words, extended Leveque solutions can be applied to lower Graetz numbers with larger flow indices as shown in Figs. 8–11.

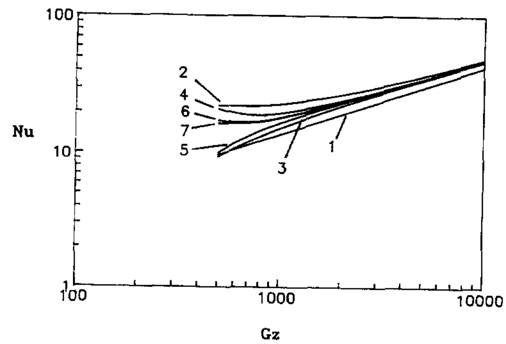

Fig. 7. Nusselt vs Graetz number for a constant wall heat flux, with $n = 0.25$, $Br = 0$ and $\beta = 0.1$: $1, 2, \ldots$, denote extended Leveque solutions with one, two,... terms, respectively.

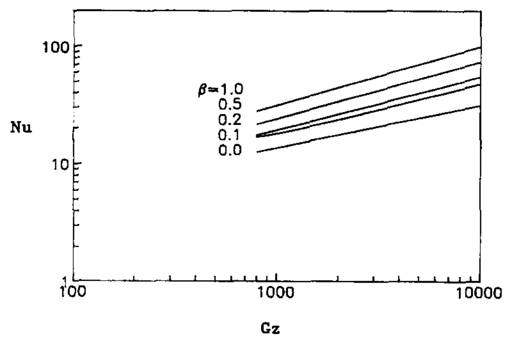

Fig. 8. Nusselt number vs Graetz number for a constant wall heat flux with $n = 0.25$, $Br = 0$.

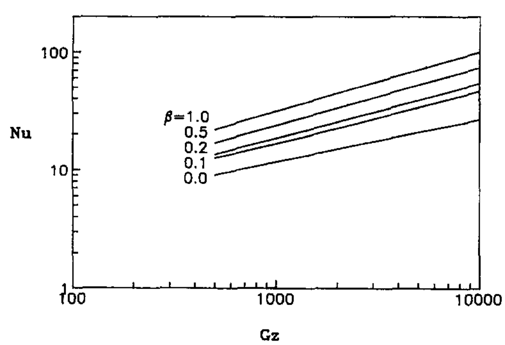

Fig. 9. Nusselt number vs Graetz number for a constant wall heat flux with $n = 0.5$, $Br = 0$.

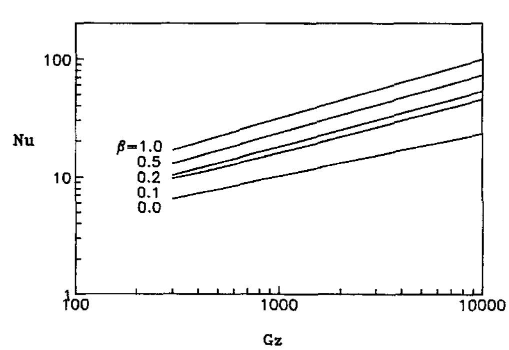

Fig. 10. Nusselt number vs Graetz number for a constant wall heat flux with $n = 1.0$, $Br = 0$.

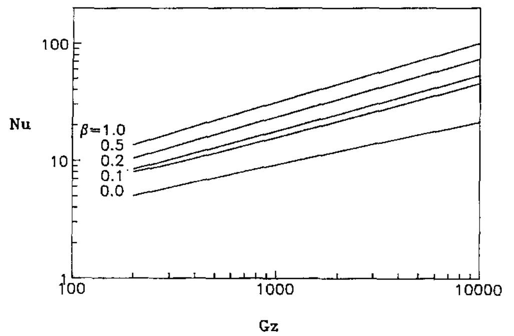

Fig. 11. Nusselt number vs Graetz number for a constant wall heat flux with $n = 2.0$, $Br = 0$.

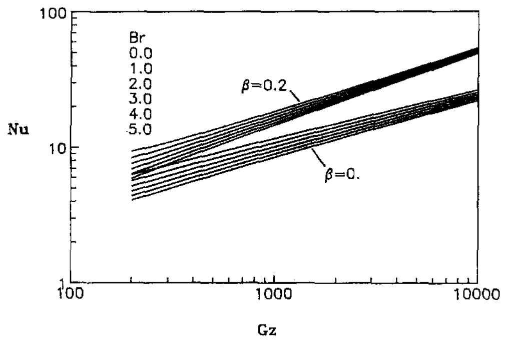

Fig. 12. Nusselt number vs Graetz number for a constant wall heat flux with $n = 0.5$, $\beta = 0.2$ and $\beta = 0$.

The effect of the slip parameter on the local Nusselt number is shown in Figs. 8–11. The Nusselt number increases with an increase in the slip parameter and the Graetz number. Figure 12 illustrates the viscous dissipation effect on the Nusselt number. The wall slip case has a higher Nusselt number as also shown in Fig. 6.

DISCUSSION AND CONCLUSION

An extended Leveque method has been applied to obtain approximate solutions of the Graetz problem for a short axial distance for power-law non-Newtonian fluids with a wall slip condition for the constant wall heat temperature and constant wall heat flux cases. The series expansion is quite different from that of the no wall slip case. The Nusselt number increases with an increase in the slip parameter and the Graetz number. The wall slip causes an increase in the Nusselt number compared to the no wall slip case.

REFERENCES

1. R. A. Mashelkar and A. Dutta, Convective diffusion in structured fluids: need for new analysis and design strategies, *Chem. Engng Sci.* **37,** 969–985 (1982).
2. Y.-H. Ju and Y.-S. Chou, Effect of wall slip on Graetz problem of power law fluid, *J. Chin. Inst. Chem. Engng* **15,** 103–109 (1984).
3. E. M. Sparrow and S. H. Lin, Laminar heat transfer in tubes under slip-flow conditions, *J. Heat Transfer* **84,** 363–369 (1962).
4. E. M. Sparrow, G. S. Brevers and L. Y. Hung, Channel and tube flow with surface mass transfer and velocity slip, *Phys. Fluids* **14,** 1312–1319 (1971).
5. R. S. Singh and R. L. Laurence, Influence of slip velocity at a membrance surface on ultrafiltration preformance, I. Channel flow system, II. Tube flow system, *Int. J. Heat Mass Transfer* **22,** 721–729 and 731–737 (1979).
6. W. Michaeli, *Extrusion Dies*, pp. 65–70. Hanser, Munich (1984).
7. R. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena*, p. 364. Wiley, New York (1960).
8. J. Newman, Extension of the Leveque solution, *J. Heat Transfer* **91,** 177–178 (1969).
9. Y. P. Shih and J. D. Tsou, Extended Leveque solutions for heat transfer to power law fluids in laminar flow in a pipe, *Chem. Engng J.* **15,** 55–62 (1978).
10. S. M. Richardson, Extended Leveque solutions for flows of power-law fluids in pipes and channels, *Int. J. Heat Mass Transfer* **22,** 1417–1423 (1979).
11. J. C. Gottifredi, O. D. Quiroga and A. F. Flores, Heat transfer to Newtonian and non-Newtonian fluids flowing in a laminar regime, *Int. J. Heat Mass Transfer* **26,** 1215–1220 (1983).
12. J. C. Gottifredi and A. F. Flores, Extended Leveque solution for heat transfer to non-Newtonian fluids in pipes and flat ducts, *Int. J. Heat Transfer* **28,** 903–908 (1985).
13. J. D. Chen and Y. H. Ju, Extended Leveque solution for heat transfer to power-law fluids in pipes, *J. Chin. Inst. Engrs* **11,** 305–307 (1988).